Math 60 # 9.6 Rationalizing Radical Expressions

Objectives
1) Rationalize a denominator containing one term
   - square root
   - higher-index root
2) Rationalize a denominator containing two terms
   - square roots only

Recall: A rational number is a number that can be written as a fraction ex: $\frac{2}{3}$, $.2$, 4, $-.\overline{6}$

and has a repeating or terminating decimal.

Key: Does not have a radical.

Recall: An irrational number is a number that cannot be written as a fraction. ex. $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, π, $\sqrt[3]{2}$, $\sqrt[3]{4}$, $\sqrt[4]{2}$

To rationalize is to re-write as an equivalent expression so that the irrational number is moved to different place.

This is a purely cosmetic change
- The entire expression is an irrational number
- When we are done, the entire expression is still the same irrational number.
- We are only changing how it looks
  - make denominator rational
  - make numerator irrational

Why? Because in higher-level classes (like calculus) we need to rearrange terms so we can simplify.

- How? We multiply by 1.

  But we will write 1 as a fraction, with the same radical in the numerator and in the denominator.

But first, let's review multiplying and notice whether the final answer is rational or irrational.

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Review

Multiply.
Identify if the answer is rational or irrational.

① $\sqrt{5} \cdot \sqrt{5}$

$= \sqrt{25}$

$= \boxed{5}$ $\boxed{\text{rational}}$

② $\sqrt[3]{2} \cdot \sqrt[3]{4}$

$= \sqrt[3]{8}$

$= \boxed{2}$ $\boxed{\text{rational}}$

③ $\sqrt[4]{2} \cdot \sqrt[4]{8}$

$= \sqrt[4]{16}$

$= \boxed{2}$ $\boxed{\text{rational}}$

④ $\sqrt[3]{2} \cdot \sqrt[3]{2}$

$= \boxed{\sqrt[3]{4}}$ $\boxed{\text{irrational}}$

4 is not a perfect cube, so this radical does not simplify.

multiplying a cube root by itself does NOT give a rational result. ☹

⑤ $3\sqrt{7} \cdot \sqrt{7}$

$= 3\sqrt{49}$

$= 3 \cdot 7$

$= \boxed{21}$ $\boxed{\text{rational}}$

⑥ $(1+\sqrt{2})(1-\sqrt{2})$

$= 1 \underbrace{- \sqrt{2} + \sqrt{2}}_{} - \sqrt{4}$ FOIL

$= 1 + 0 - 2$

$= \boxed{-1}$ $\boxed{\text{rational}}$

This is a difference of squares!
$(a+b)(a-b) = a^2 - b^2$.

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(7) $(5-\sqrt{3})(5+\sqrt{3})$

$= 25 \underbrace{+5\sqrt{3} - 5\sqrt{3}} - \sqrt{9}$

$= 25 \quad + 0 \quad - 3$

$= \boxed{22}$ $\boxed{\text{rational}}$

(8) $(5-\sqrt{3})^2$

$= (5-\sqrt{3})(5-\sqrt{3})$ FOIL

$= 25 \underbrace{-5\sqrt{3} - 5\sqrt{3}} - \sqrt{9}$

$= 25 \quad -10\sqrt{3} \quad - 3$ combine like radicals

$= \boxed{22 - 10\sqrt{3}}$ $\boxed{\text{irrational}}$

When there are 2 terms, multiplying by itself does NOT give a rational result. ☹.

(9) $\sqrt{3}(5-\sqrt{3})$

$= 5\sqrt{3} - \sqrt{9}$ distribute

$= \boxed{5\sqrt{3} - 3}$ $\boxed{\text{irrational}}$

When there are 2 terms, multiplying by only the radical does NOT give a rational result. ☹.

Definition: When two binomials multiply to give a difference of squares, we say the binomials are conjugates of each other.

Example: $(1-\sqrt{2})$ is the conjugate of $(1+\sqrt{2})$ and vice-versa, because $(1-\sqrt{2})(1+\sqrt{2}) = 1^2 - (\sqrt{2})^2$

Example: $(5+\sqrt{3})$ is the conjugate of $(5-\sqrt{3})$ and vice-versa, because $(5+\sqrt{3})(5-\sqrt{3}) = 5^2 - (\sqrt{3})^2$

Find the conjugate of each expression.

(10) $6-\sqrt{3}$ conjugate is $\boxed{6+\sqrt{3}}$

(11) $3+\sqrt{5}$ conjugate is $\boxed{3-\sqrt{5}}$

(12) $2\sqrt{3} - 4\sqrt{5}$ conjugate is $\boxed{2\sqrt{3}+4\sqrt{5}}$

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Rationalize each denominator.

(13) $\frac{1}{\sqrt{5}}$ notice: denominator is irrational

multiply by $1 = \frac{\sqrt{5}}{\sqrt{5}}$

$$= \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{\sqrt{5}}{\sqrt{25}}$$

$$= \boxed{\frac{\sqrt{5}}{5}}$$

notice: denominator is rational.
The denominator has been rationalized.

(14) $\frac{\sqrt{3}}{\sqrt{32}}$

simplify $\sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$

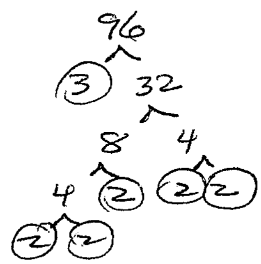
$$= \frac{\sqrt{3}}{4\sqrt{2}}$$

multiply by $1 = \frac{\sqrt{2}}{\sqrt{2}}$

$$= \frac{\sqrt{3}}{4\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6}}{4 \cdot 2}$$

$$= \boxed{\frac{\sqrt{6}}{8}}$$



Note: While it is legal to use $\frac{\sqrt{32}}{\sqrt{32}}$, it's ugly:

$$\frac{\sqrt{3}}{\sqrt{32}} \cdot \frac{\sqrt{32}}{\sqrt{32}} = \frac{\sqrt{96}}{32} = \frac{4\sqrt{6}}{32} = \frac{\sqrt{6}}{8}.$$

Math 60 9.6 Assume all variables are positive.

(15) $\dfrac{3}{2\sqrt{5x}}$

multiply by $1 = \dfrac{\sqrt{5x}}{\sqrt{5x}}$

$= \dfrac{3}{2\sqrt{5x}} \cdot \dfrac{\sqrt{5x}}{\sqrt{5x}}$

$= \dfrac{3\sqrt{5x}}{2 \cdot 5x}$

$= \boxed{\dfrac{3\sqrt{5x}}{10x}}$

(16) $\dfrac{5}{\sqrt[3]{2}}$

Remember: $\sqrt[3]{2} \cdot \sqrt[3]{2} = \sqrt[3]{4}$ is not rational!

We need $\sqrt[3]{2^3}$ so we are missing $\sqrt[3]{2^2} = \sqrt[3]{4}$

$= \dfrac{5}{\sqrt[3]{2}} \cdot \dfrac{\sqrt[3]{4}}{\sqrt[3]{4}}$

$= \dfrac{5\sqrt[3]{4}}{\sqrt[3]{8}}$

$= \boxed{\dfrac{5\sqrt[3]{4}}{2}}$

(17) $\sqrt[3]{\dfrac{5}{12}}$

Because this fraction has a radical on its denominator, its denom is not rationalized.

$= \dfrac{\sqrt[3]{5}}{\sqrt[3]{12}}$

12

6 (2)

(2) (3)

$\sqrt[3]{12} = \sqrt[3]{2^2 \cdot 3}$

need $\sqrt[3]{2}$ need $\sqrt[3]{3^2}$

$= \dfrac{\sqrt[3]{5}}{\sqrt[3]{2^2 \cdot 3}} \cdot \dfrac{\sqrt[3]{2 \cdot 3^2}}{\sqrt[3]{2 \cdot 3^2}}$

$$= \frac{\sqrt[3]{5\cdot 2\cdot 3^2}}{\sqrt[3]{2^3\cdot 3^3}}$$

$$= \frac{\sqrt[3]{90}}{2\cdot 3}$$

because there are no perfect cubes in the prime factors of 90, we cannot simplify $\sqrt[3]{90}$.

$$= \boxed{\frac{\sqrt[3]{90}}{6}}$$

NOTE: If you have the correct denominator but the wrong numerator, check the index of the radical. You probably didn't use the correct radicand (and assumed the correct result for denominator.)

(18) $\dfrac{10}{\sqrt[4]{2z^2}}$ Assume all variables are positive.

missing 2^3 missing z^2

notice index 4 $\Rightarrow$ We need perfect 4th powers in denom

$\sqrt[4]{2^4\cdot z^4}$

$$= \frac{10}{\sqrt[4]{2z^2}} \cdot \frac{\sqrt[4]{2^3\cdot z^2}}{\sqrt[4]{2^3\cdot z^2}}$$

$$= \frac{10\sqrt[4]{8z^2}}{\sqrt[4]{2^4 z^4}}$$

$$= \frac{10\sqrt[4]{8z^2}}{2z}$$

Notice $\frac{10}{2} = 5$ reduces.

$$= \boxed{\frac{5\sqrt[4]{8z^2}}{z}}$$

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19 $\dfrac{11}{6-\sqrt{3}}$

Notice: Two terms in denominator.

Remember that multiplying a binomial by its conjugate gives a rational number.

The conjugate of $6-\sqrt{3}$ is $6+\sqrt{3}$.

$$= \frac{11}{(6-\sqrt{3})} \cdot \frac{(6+\sqrt{3})}{(6+\sqrt{3})}$$ multiply by $1 = \dfrac{6+\sqrt{3}}{6+\sqrt{3}}$

$$= \frac{11(6+\sqrt{3})}{36+6\sqrt{3}-6\sqrt{3}-\sqrt{9}}$$

← Leave 11 outside because it's rational. We hope to reduce.

↙ FOIL denom

$$= \frac{11(6+\sqrt{3})}{36-3}$$

$$= \frac{11(6+\sqrt{3})}{33}$$ notice $\dfrac{11}{33}$ reduces to $\dfrac{1}{3}$ $(\div 11)$

$$= \boxed{\frac{6+\sqrt{3}}{3}}$$ also equal to $\dfrac{6}{3} + \dfrac{\sqrt{3}}{3} = \boxed{2+\dfrac{\sqrt{3}}{3}}$

20 $\dfrac{\sqrt{3}}{3+\sqrt{5}}$

The conjugate of $3+\sqrt{5}$ is $3-\sqrt{5}$

$$= \frac{\sqrt{3}}{(3+\sqrt{5})} \cdot \frac{(3-\sqrt{5})}{(3-\sqrt{5})}$$

— dist numerator (because $\sqrt{\ }$ outside is irrational => no chance to reduce)

— FOIL denom

$$= \frac{3\sqrt{3}-\sqrt{15}}{9+3\sqrt{5}-3\sqrt{5}-\sqrt{25}}$$

$$= \frac{3\sqrt{3}-\sqrt{15}}{9-5} = \boxed{\frac{3\sqrt{3}-\sqrt{15}}{4}}$$

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21 $\dfrac{\sqrt{5}-2}{\sqrt{3}-\sqrt{5}}$

The conjugate of $\sqrt{3}-\sqrt{5}$ is $\sqrt{3}+\sqrt{5}$

$$= \frac{(\sqrt{5}-2)}{(\sqrt{3}-\sqrt{5})} \cdot \frac{(\sqrt{3}+\sqrt{5})}{(\sqrt{3}+\sqrt{5})}$$

← FOIL numerator

← FOIL denominator

$$= \frac{\sqrt{15}+5-2\sqrt{3}-2\sqrt{5}}{3+\sqrt{15}-\sqrt{15}-\sqrt{25}}$$

$$= \frac{5-2\sqrt{3}-2\sqrt{5}+\sqrt{15}}{3-5}$$

It's not required that you write the radicals in order, but it's nicer if you do.

$$= \frac{5-2\sqrt{3}-2\sqrt{5}+\sqrt{15}}{-2}$$

negative in denominator is not simplified

$$= \frac{-(5-2\sqrt{3}-2\sqrt{5}+\sqrt{15})}{2}$$

move negative to numerator

$$= \boxed{\frac{-5+2\sqrt{3}+2\sqrt{5}-\sqrt{15}}{2}}$$

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22.

$$\frac{4\sqrt{3}+5\sqrt{2}}{2\sqrt{3}-4\sqrt{5}} \cdot \frac{2\sqrt{3}+4\sqrt{5}}{2\sqrt{3}+4\sqrt{5}}$$

$$= \frac{8\cdot 3 + 16\sqrt{15} + 10\sqrt{6} + 20\sqrt{10}}{4\cdot 3 - 16\cdot 5}$$

$$= \frac{24 + 16\sqrt{15} + 10\sqrt{6} + 20\sqrt{10}}{12-80}$$

$$= \frac{24 + 16\sqrt{15} + 10\sqrt{6} + 20\sqrt{10}}{-68}$$

$$= \frac{+2\left(12 + 8\sqrt{15} + 5\sqrt{6} + 10\sqrt{10}\right)}{-2(34)}$$

$$= \frac{-\left(12 + 8\sqrt{15} + 5\sqrt{6} + 10\sqrt{10}\right)}{34}$$

$$= \boxed{\frac{-12 - 8\sqrt{15} - 5\sqrt{6} - 10\sqrt{10}}{34}}$$